Twisted cohomology for hyperbolic three manifolds.

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Abstract

For a complete hyperbolic three manifold M, we consider the representations of $\pi_1(M)$ obtained by composing a lift of the holonomy with complex finite dimensional representations of $\mathrm{SL}(2, \mathbf{C})$. We prove a vanishing result for the cohomology of M with coefficients twisted by these representations, using techniques of Matsushima-Murakami. We give some applications to local rigidity.

Let M be an orientable complete hyperbolic three manifold. The holonomy representation of the complete hyperbolic structure

$$\mathrm{Hol}\colon \pi_1(M) \to \mathrm{Isom}^+ \mathbf{H}^3 \cong \mathrm{PSL}(2, \mathbf{C}),$$

can be lifted to a representation $\widetilde{\mathrm{Hol}}\colon \pi_1(M, p) \to \mathrm{SL}(2, \mathbf{C})$ (see for instance [9]). Moreover, there is a one-to-one correspondence between these lifts and spin structures on M.

Suppose that V is a finite *complex* representation of $\mathrm{SL}(2, \mathbf{C})$. Composing this representation with $\widetilde{\mathrm{Hol}}$, we obtain a representation $\rho\colon \pi_1(M) \to \mathrm{SL}(V)$, and hence we can consider the associated flat vector bundle

$$E_\rho = \widetilde{M} \times_{\pi_1(M)} V.$$

The space of E_ρ-valued differential forms on M will be denoted by $\Omega^*(M; E_\rho)$. A $\mathrm{SU}(2)$-invariant hermitian product on V induces a hermitian metric on E_ρ, and also a hermitian product on the space of forms $\Omega^*(M; E_\rho)$. In particular we may talk about L^2-forms of $\Omega^*(M; E_\rho)$, as those which are square summable.

We say that ρ has *trivial factors* if in the decomposition of V as direct sum of irreducible $\mathrm{SL}(2, \mathbf{C})$-representations, there is the trivial representation $\mathbf{C}$. Having a trivial factor is equivalent to the existence of a nonzero vector in V which is $\rho(\pi_1(M))$-invariant.

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Theorem 0.1 ([22])**.** *Let M be a complete hyperbolic 3-manifold. If ρ has no trivial factors, then every closed L^2-form in $\Omega^1(M; E_\rho)$ is exact.*

This theorem is due to Raghunathan [22] (cf. [5]), who proves it only in the compact case, but the same proof works in general. His theorem applies to other representations and other lattices, and we think that it is worth to give a shorter proof in our situation.

This theorem has the following cohomology vanishing result as a corollary.

Corollary 0.2 ([22])**.** *Let M be a* closed *hyperbolic manifold. If ρ has no trivial factors, then $H^*(M; E_\rho) = 0$.*

We say that a hyperbolic three manifold M is *topologically finite* if it is the interior of a compact manifold $\overline{M}$. This is equivalent to say that $\pi_1(M)$ is finitely generated, by the proof of Marden's conjecture [1, 8].

Corollary 0.3. *Let M be a complete hyperbolic 3-manifold that is topologically finite. If V has no trivial factors, then the inclusion $\partial\overline{M} \subset \overline{M}$ induces an injection.*

$$H^1(M; E_\rho) \hookrightarrow H^1(\partial\overline{M}; E_\rho)$$

Proof. The space $H^1(\overline{M}, \partial\overline{M}; E_\rho)$ can be identified with the first cohomology group of E_ρ-valued forms with compact support. Hence, Theorem 0.1 implies that the map $H^1(\overline{M}, \partial\overline{M}; E_\rho) \to H^1(\overline{M}; E_\rho)$ induced by the inclusion is zero. The corollary now follows from the long exact sequence of the pair. □

It is well known that for every positive integer $n \geq 1$ there exists a unique, up to isomorphism, complex irreducible representation of $\mathrm{SL}(2, \mathbf{C})$ of dimension n, which is isomorphic to the $(n-1)$-symmetric power of the standard representation of $\mathrm{SL}(2, \mathbf{C})$.

If M is a complete hyperbolic 3-manifold of finite volume with a single cusp, then $\partial\overline{M}$ is a torus. An analysis of the group $H^0(\partial\overline{M}; E_\rho)$ (see Section 2), together with an Euler characteristic argument and Poincaré duality, gives the following theorem.

Theorem 0.4. *Let M be a complete hyperbolic 3-manifold of finite volume. If M has a single cusp and V is the irreducible complex representation of $\mathrm{SL}(2, \mathbf{C})$ of dimension $2k$, with $k \geq 1$, then $H^1(\partial\overline{M}; E_\rho) = 0$. In particular*

$$H^*(M; E_\rho) = 0.$$

Notice that this theorem applies to hyperbolic knot exteriors in S^3.

Theorem 0.1 has applications to infinitesimal rigidity as well. Let's write ρ_n to denote the representation ρ defined above when V is the n-dimensional irreducible representation:

$$\rho_n \colon \pi_1(M) \to \mathrm{SL}(n, \mathbf{C}).$$

The space of infinitesimal deformations of ρ_n is isomorphic to $H^1(M; E_{\mathrm{Ad}\circ\rho_n})$, where $\mathrm{Ad}\colon \mathrm{SL}(n,\mathbf{C}) \to \mathrm{Aut}(\mathfrak{sl}(n,\mathbf{C}))$ is the adjoint representation.

The following theorem is an infinitesimal rigidity result for ρ_n in $\mathrm{SL}(n,\mathbf{C})$ relative to the boundary. Its proof uses the decomposition of representation $\mathfrak{sl}(n,\mathbf{C})$ into irreducible factors, and will be given in Section 3.

Theorem 0.5. *Let M be a complete, hyperbolic, orientable 3-manifold that is topologically finite. If $\partial\overline{M}$ is the union of k tori and l surfaces of genus $g_1,\ldots,g_l \geq 2$,, and $n \geq 2$, then*

$$H^1(M; E_{\mathrm{Ad}\circ\rho_n}) \cong \mathbf{C}^{k(n-1)+\sum(g_i-1)(n^2-1)}.$$

In particular, if M is closed then $H^1(M; E_{\mathrm{Ad}\circ\rho_n}) = 0$. In addition, all nontrivial elements in $H^1(M; E_{\mathrm{Ad}\circ\rho_n})$ are nontrivial in $\partial\overline{M}$ and have no L^2 representative.

When $n = 2$, this is Weil's infinitesimal rigidity in the compact case, and Garland's L^2-infinitesimal rigidity in the noncompact case. This has been generalised to cone three manifolds by Hodgson-Kerckhoff [15], Weiss [26] and Bromberg [6].

Let $X(M, \mathrm{SL}(n,\mathbf{C}))$ be the variety of characters of $\pi_1(M)$ in $\mathrm{SL}(n,\mathbf{C})$. The character of ρ_n is denoted by χ_{ρ_n}. From the previous theorem and standard results on the variety of characters, we deduce:

Theorem 0.6. *Let M be a topologically finite, hyperbolic, orientable 3-manifold as in Theorem 0.5. If $n \geq 2$, then the character χ_{ρ_n} is a smooth point of $X(M, \mathrm{SL}(n,\mathbf{C}))$ with tangent space $H^1(M; E_{\mathrm{Ad}\circ\rho_n})$.*

For $n = 2$, this is Theorem 8.44 of Kapovich [17].

This paper is organized as follows. Section 1 is devoted to prove Theorem 0.1. In the first part of that section we explain the results by Matsushima and Murakami [20] concerning the theory of harmonic forms on a locally symmetric manifold. The main result there is a Weitzemböck formula for the Laplacian. We have thought convenient to review this material with a slightly more modern language than the original one given in [20]. Although it does not bring in a new conceptual approach, we hope the exposition given here will be more accessible for the non-expert. The second part of that section gives the proof of Theorem 0.1. We will do this by establishing that certain endomorphism H involved in the Weitzemböck formula is positively defined. From this, a strong positivity condition of the Laplacian can be easily established; it will allow us to to apply a theorem of Andreotti and Vesentini [2], and hence establish Theorem 0.1. (Strictly speaking, Andreotti and Vesentini work with complex manifolds; here we use the adaptation of Garland [11, Thm. 3.22].) The positivity of the endomorphism H is done by Raghunathan in a more general setting in [22], but here we give a simple proof in our situation.

In section 2 we compute the cohomology of the ends, which is going to be used to the cohomology of M, in view of Corollary 0.3. Section 3.deals with applications to infinitesimal and local rigidity, in particular we prove Theorems 0.5 and 0.6. In section 4 we discuss some properties of lifts of representations and prove Corollary 0.4.

Appendix A reviews some results about principal bundles that are required in Section 1.

1 Proof of the cohomology vanishing theorem

The aim of this section is to prove Theorem 0.1. Although this is not directly a cohomology vanishing result, it implies Corollary 0.2. We start with some general considerations.

Given a vector bundle E over a manifold M, the cohomology groups $H^p(M;E)$ can be defined as the cohomology of the sheaf of germs of sections of E. A way to compute a resolution of this sheaf -and hence compute its cohomology- consists in introducing a flat connection on E, and then consider the associated covariant exterior differential D on the space of E-valued forms.

The classical Hodge's theorem states that in a *closed* Riemannian manifold M, the space of degree p harmonic forms is canonically isomorphic to the group $H^p(M;\mathbf{R})$. It is not difficult to generalize this result to the case of a flat vector bundle E, and conclude that there is also a canonical isomorphism between $H^p(M;E)$ and the space of degree p E-valued harmonic forms. Therefore, we can obtain information about the cohomology of a closed Riemannian manifold by inspecting its Laplacian.

A well known result due to Andreotti and Vesentini [2] gives a criterion for the vanishing of the cohomology groups. As said in the introduction, Andreotti and Vesentini work with complex manifolds. Here we will use the adaptation of Garland [11, Thm. 3.22].

Theorem 1.1 (Andreotti-Vesentini, Garland)**.** *Suppose that M is complete. Assume that there exists $c > 0$ such that for every $\alpha \in \Omega^p(M;E)$ with compact support $(\Delta\alpha,\alpha) \geq c(\alpha,\alpha)$, where $(\,,)$ denotes the inner product on the space of E-valued forms. Then every square-integrable closed p-form is exact.*

1.1 Review of harmonic forms on a locally symmetric manifold

The aim of this section is to obtain a Weitzenböck formula for a locally symmetric manifold. The material presented here is almost entirely based on Matsushima and Murakami [20].

Let G be a connected semi-simple Lie group and $K < G$ a maximal compact subgroup of G. The respective Lie algebras are denoted by $\mathfrak{g}$ and $\mathfrak{k}$, with the convention that they are the Lie algebras of left invariant vector fields on G and K, respectively.

Let B denote the Killing form of $\mathfrak{g}$. We recall that it is defined by

$$B(V, W) = \operatorname{tr}(\operatorname{ad}_V \circ \operatorname{ad}_W),$$

for $V, W \in \mathfrak{g}$. Cartan's criterion implies that B is non-degenerated if, and only if, $\mathfrak{g}$ is semisimple. In that case, we have a canonical decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, where $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{k}$ respect to B. This decomposition satisfies the following properties: B is negatively defined on $\mathfrak{k}$; B positively defined on $\mathfrak{m}$; $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$; and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

The Killing form defines a pseudo-Riemannian metric on G, which is invariant by the action of G by right translations, and is positively (resp. negatively) defined on $\mathfrak{m}$ (resp. $\mathfrak{k}$). Therefore, the Killing form defines a Riemannian metric on the homogeneous space $X = G/K$. Note that G acts on the left on X by orientation preserving isometries.

Let Γ be a discrete subgroup of G that acts freely on X. Since Γ acts by isometries, the quotient $M = \Gamma\backslash X$ is a Riemannian manifold. It is said that M is a *locally symmetric manifold*.

For our purposes, it will be convenient to regard the universal covering $X \to M$ as a principal bundle over M with structure group Γ. We follow the convention that the action of the structure group of a principal bundle is on the right. Hence we only need to convert the action of Γ into a right action (if $g \in \Gamma$, then $x \cdot g = g^{-1} \cdot x$, for $x \in X$). We will also regard X as a flat bundle.

Consider the G-principal bundle $P = X \times_\Gamma G$ over M (see Appendix A for notation) endowed with the flat connection induced from the trivial connection of the product $X \times G$. We can embed X on P using the section $X \to X \times G$ whose second coordinate is constant and equal to the identity element. We can think of X as a reduction of the structure group. Obviously, the horizontal leaves of X are also horizontal leaves of P, so the connection on P is reducible to X.

On the other hand, the principal bundle P has a canonical reduction of its structure group from G to K. In order to get such a reduction, consider the embedding $i \colon G \hookrightarrow X \times G$ given by $i(g) = (gK, g)$. The image of G by this embedding is invariant by the bundle action of K, so it defines an embedding $\Gamma\backslash G \hookrightarrow X \times_\Gamma G$, which will be also denoted by i. Therefore, $Q = i(\Gamma\backslash G)$ is a reduction of the structure group.

The connection defined on P is not reducible to Q, because its horizontal distribution is not tangent to Q (a curve on $X \times G$ whose second component is constant, gives an horizontal curve on P; hence, if the horizontal distribution were tangent to Q, this curve would be contained in Q, and this does not

happen). Nevertheless, since the action of K on $\mathfrak{g}$ respects the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, we can state the following.

Proposition 1.2. *Let $\eta \in \Omega^1(P; \mathfrak{g})$ be the connection form of the connection defined on P above. Put $\eta = \eta_{\mathfrak{m}} + \eta_{\mathfrak{k}}$, where $\eta_{\mathfrak{m}}$ and $\eta_{\mathfrak{k}}$ are the $\mathfrak{m}$ and $\mathfrak{k}$ components of η respectively. Then, the restriction of $\eta_{\mathfrak{k}}$ to Q is a connection form on Q.*

Observation. We can identify $\mathfrak{g}$ with the space of vector fields on $\Gamma\backslash G$ that are projection of left invariant vector fields on G. In what follows, we will tacitly do this identification.

Let $\omega \in \Omega^1(\Gamma\backslash G; \mathfrak{g})$ be the left Maurer-Cartan form of G. It is easily checked that $i^*(\eta) = \omega$. Hence, if we decompose $\omega = \omega_{\mathfrak{m}} + \omega_{\mathfrak{k}}$ into the $\mathfrak{m}$-component and the $\mathfrak{k}$-component, $\omega_{\mathfrak{k}}$ is the connection form of the connection defined on $\Gamma\backslash G$, and the horizontal distribution is given by $\mathfrak{m}$.

Consider a finite linear representation $\rho\colon G \longrightarrow \mathrm{Aut}(V)$, and the associated vector bundle $E = X \times_\Gamma V$ (note that E is canonically identified with $P \times_G V$ and $Q \times_K V$).

The flat connection on P defines an exterior covariant differential d_ρ on the space $\Omega^*(M; E)$. Via the canonical isomorphism between $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ and $\Omega^*(M; E)$, we can transfer the operator d_ρ to an operator D_ρ, in such a way that this isomorphism is a chain complex isomorphism. If we denote by D the exterior covariant differential defined by the connection $\omega_{\mathfrak{k}}$ on Q, then the relation between D and D_ρ is given by the following proposition.

Proposition 1.3. *Let α be a form in $\Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$. We have the following decomposition*

$$D_\rho \alpha = D\alpha + T\alpha,$$

where $T\alpha = \rho(\omega_m) \wedge \alpha$.

Proof. On P the differential covariant is given by $d\alpha + \rho(\eta) \wedge \alpha$ (see Proposition A.2). Hence, if we transfer it to Q via i, we get $D_\rho \alpha = d\alpha + \rho(i^*\eta) \wedge \alpha$, and the proposition follows from the fact that $i^*\eta = \omega$. $\square$

Let's fix an orientation on $\mathfrak{k}$ and $\mathfrak{m}$, and take an orthonormal basis for $\mathfrak{g}$, $(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$, such that $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ are positively oriented orthonormal bases for $\mathfrak{k}$ and $\mathfrak{m}$, respectively. Here, orthonormality means that

$$B(X_i, X_j) = -\delta_{ij} \quad B(Y_i, Y_j) = \delta_{ij}, \quad B(X_i, Y_j) = 0.$$

Notation. We will follow the following conventions. Let V be a finite dimensional vector space. If $e_1, \ldots, e_n$ is a basis for V, then its dual basis will be denoted by $e^1, \ldots, e^n \in V^*$, with $e^i(e_j) = \delta_{ij}$. If $A \in \bigotimes^r V^*$ is an

r-times covariant tensor, then its components relative to the basis defined by $e^1, \ldots, e^n$ will be denoted by $A_{i_1,\ldots,i_r}$. Concerning the exterior product on $\bigwedge^* V^*$, we will follow the convention such that $e^1 \wedge \cdots \wedge e^n$ is the determinant. We will also use Einstein notation. Hence, given $\alpha \in \bigwedge^r V^*$, we have $\alpha = \alpha_{i_1,\ldots,i_r} e^{i_1} \otimes \cdots \otimes e^{i_r}$, where $\alpha_{i_1,\ldots,i_r}$ are scalars satisfying $\alpha_{i_{\sigma(1)},\ldots,i_{\sigma(r)}} = \operatorname{sgn}(\sigma)\alpha_{i_1,\ldots,i_r}$, for any permutation $\sigma \in \Sigma_r$. Then we also have

$$\alpha = \sum_{1 \le i_1 < \cdots < i_r \le r} \alpha_{i_1,\ldots,i_r} e^{i_1} \wedge \cdots \wedge e^{i_r} = \frac{1}{r!}\alpha_{i_1,\ldots,i_r} e^{i_1} \wedge \cdots \wedge e^{i_r}.$$

From now on, all the tensors will be written in the basis of $\mathfrak{g}$ given by $\{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$.

Proposition 1.4. *For $\alpha \in \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$, the operators D and T are given by the following equations.*

$$(D\alpha)_{i_1,\ldots,i_{r+1}} = \sum_{k=1}^{r+1} (-1)^{k+1} Y_{i_k} \alpha_{i_1,\ldots,\widehat{i_k},\ldots,i_{r+1}} \tag{1}$$

$$(T\alpha)_{i_1,\ldots,i_{r+1}} = \sum_{k=1}^{r+1} (-1)^{k+1} \rho(Y_{i_k}) \alpha_{i_1,\ldots,\widehat{i_k},\ldots,i_{r+1}}. \tag{2}$$

Proof. Put $\alpha = \frac{1}{r!}\alpha_{i_1,\ldots,i_r} Y^{i_1} \wedge \cdots \wedge Y^{i_r}$. By definition, $D\alpha$ is the horizontal component of $d\alpha$. It is immediate that dY^k has no horizontal component: $dY^k(Y_i, Y_j) = Y^k([Y_i, Y_j]) = 0$. Hence, $D\alpha = \frac{1}{r!} Y_j \alpha_{i_1,\ldots,i_r} \otimes Y^j \wedge Y^{i_1} \wedge \cdots \wedge Y^{i_r}$. Rearranging the indices we get equation 1. The other equation follows immediately from the definition of T. ☐

Let us define the forms $\Omega_K = X^1 \wedge \cdots \wedge X^n$ and $\Omega_M = Y^1 \wedge \cdots \wedge Y^m$. It is clear that these forms are independent of the orthonormal bases chosen. Hence, Ω_K and Ω_M are well defined forms on $\Gamma\backslash G$. Note that Ω_K is vertical and Ω_M is horizontal, and both are right K-invariant (it is a consequence of the fact the right action of K on $\mathfrak{g}$ leaves both the Killing form and the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ invariant). Observe that Ω_M defines a volume form on M, which is compatible with the metric structure of M.

Next we want to define an inner product on the fibers of E. In order to do that, fix a K-invariant inner product $\langle\, , \rangle_V$ on V, and use it to define a metric on the fibers of $E = Q \times_K V$. Then define an inner product on $\Omega^*(M; E)$ as usual: if $\alpha, \beta \in \Omega^*(M; E)$ then

$$(\alpha, \beta) = \int_M \langle \alpha(x), \beta(x) \rangle_x \Omega_M,$$

where $\langle\, , \rangle_x$ is the inner product defined on the fiber E_x, and Ω_M is interpreted as a form on M. On the other hand, we can define the inner product

of two forms $\tilde{\alpha}, \tilde{\beta} \in \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ by

$$(\tilde{\alpha}, \tilde{\beta}) = \frac{1}{\mu(K)} \int_{\Gamma\backslash G} \langle \tilde{\alpha}(u), \tilde{\beta}(u)\rangle_u \Omega_K \wedge \Omega_M,$$

where $\langle\, ,\rangle_u$ is the inner product on $\bigwedge^r H^* \otimes V$ induced by the the Killing form, and the inner product on V, and $\mu(K) = \int_K \Omega_K$ the volume of K. Proposition A.4, gives the relation between these two products.

Proposition 1.5. *The canonical isomorphism between $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ and $\Omega^*(M; E)$ is an isometry.*

Using the Hodge dual operator on the horizontal bundle

$$*\colon \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K \to \Omega^{m-r}_{\mathrm{Hor}}(\Gamma\backslash G; V)^K,$$

we can give a characterization of the formal adjoint of the operators D and T.

Proposition 1.6. *Let $\alpha \in \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ with compact support. Then,*

$$D^*\alpha = (-1)^r *^{-1} D * \alpha \tag{3}$$

$$T^*\alpha = (-1)^{r-1} *^{-1} \overline{\rho(\omega)}^t \wedge (*\alpha) \tag{4}$$

Proof. We want to use Proposition A.5. We claim that

$$\int_P D\alpha \wedge \beta \wedge \Omega_K = (-1)^r \int_P \alpha \wedge D\beta \wedge \Omega_K,$$

for α and β forms of $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)$ with compact support of degree $r-1$ and $m-r$ respectively. Indeed, since $D\alpha$ is the horizontal component of $d\alpha$, we have $D\alpha \wedge \Omega_K = d\alpha \wedge \Omega_K$. Then,

$$d(\alpha \wedge \beta \wedge \Omega_K) = d\alpha \wedge \beta \wedge \Omega_K + (-1)^{r-1}\alpha \wedge d\beta \wedge \Omega_K,$$

for Ω_K being closed. Therefore, by Stokes' theorem we get the equality we wanted to prove. Now, Proposition A.5 gives Formula (3).

Now, let us prove (4). By Proposition A.5, it suffices to prove that

$$(\rho(\omega) \wedge \alpha) \wedge \beta = (-1)^{r-1}\alpha \wedge (\rho(\omega)^* \wedge \beta).$$

If we take an orthonormal basis for V, then α and β are column vectors of forms of degree $r-1$ and $m-r$ respectively, and $\rho(\omega)$ a matrix of one forms. Hence, in this basis $(\rho(\omega) \wedge \alpha) \wedge \beta$ is $(\rho(\omega)\alpha)^t \overline{\beta}$, but $(\rho(\omega)\alpha)^t \beta = (-1)^{r-1}\alpha^t \rho(\omega)^t \overline{\beta}$, as we wanted to prove.

□

A similar proof of Proposition 1.4, using the formulae found in the previous proposition, gives the following.

Proposition 1.7. *For $\alpha \in \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ with compact support, the operators D^* and T^* are given by the following equations.*

$$(D^*\alpha)_{i_1,\dots,i_{r-1}} = \sum_{k=1}^{m} -Y_k\alpha_{k,i_1,\dots,i_{r-1}} \tag{5}$$

$$(T^*\alpha)_{i_1,\dots,i_{r-1}} = \sum_{k=1}^{m} \rho(Y_k)\alpha_{k,i_1,\dots,i_{r-1}}. \tag{6}$$

Lemma 1.8. *If the inner product on V is symmetric respect to the action of $\mathfrak{m}$, then the operator $S = TD^* + T^*D + DT^* + D^*T$ is zero for every form with compact support.*

Before proving the lemma, we need the following result.

Lemma 1.9. *For every function f with compact support, and $Y \in \mathfrak{g}$,*

$$\int_{\Gamma\backslash G} (Yf)\Omega_M \wedge \Omega_K = 0.$$

Proof. Since Y is an infinitesimal isometry we have $L_Y(f\Omega_M \wedge \Omega_K) = (Yf)\Omega_M \wedge \Omega_K$. On the other hand, the formula $L_Y = i_Y \circ d + d \circ i_Y$ gives $L_Y(f\Omega_M \wedge \Omega_K) = d(i_Y f\Omega_M \wedge \Omega_K)$, and Stokes' theorem implies

$$0 = \int_{\Gamma\backslash G} L_Y(f\Omega_M \wedge \Omega_K) = \int_{\Gamma\backslash G} (Yf)\Omega_M \wedge \Omega_K,$$

as we wanted to prove. ☐

Proof of Lemma 1.8. Since S is a self-adjoint operator, $S = 0$ if, and only if, $(S\alpha, \alpha) = 0$ for every α with compact support. Let's take $\alpha \in \Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ with compact support. We must show that

$$(S\alpha, \alpha) = (D\alpha, T\alpha) + (T\alpha, D\alpha) + (D^*\alpha, T^*\alpha) + (T^*\alpha, D^*\alpha) = 0.$$

Observe that it suffices to prove that $(D\alpha, T\alpha) + (D^*\alpha, T^*\alpha) = 0$. Moreover, using the $\mathfrak{m}$-symmetry of the inner product and the fact that the Hodge $*$ operator is an isometry, we must prove $(D\alpha, T\alpha) + (D(*\alpha), T(*\alpha)) = 0$. Let's compute $(D\alpha, T\alpha)$. Put $\alpha = \alpha_{i_1,\dots,i_r} \otimes Y^{i_1} \wedge \cdots \wedge Y^{i_r}$. If we use the expression of D and T given in Proposition 1.4, we see that $(D\alpha, T\alpha)$ is the sum of terms of the form

$$(-1)^{i+j} \int_{\Gamma\backslash G} \langle Y_{i_j}\alpha_{i_1,\dots,\widehat{i_j},\dots,i_{r+1}}, \rho(Y_{i_k})\alpha_{i_1,\dots,\widehat{i_k},\dots,i_{r+1}}\rangle_V \, d\mu_G.$$

It is convenient to group the summands according to whether the avoided sub-indices $\widehat{i_j}$ and $\widehat{i_k}$ are equal or not. Therefore, one term is a sum of factors of the form

$$\int_{\Gamma\backslash G} \langle Y_j\alpha_{i_1,\dots,i_r}, \rho(Y_j)\alpha_{i_1,\dots,i_r}\rangle_V \, d\mu_G, \quad j \notin \{i_1,\dots,i_r\},$$

and the rest is a sum of terms of the form

$$(-1)^{j+k}\int_{\Gamma\backslash G}\langle Y_{i_j}\alpha_{i_1,\ldots,\widehat{i_j},\ldots,i_k,\ldots,i_r},\rho(Y_{i_k})\alpha_{i_1,\ldots,i_j,\ldots,\widehat{i_k},\ldots,i_r}\rangle d\mu_G, \tag{7}$$

with $i_j \neq i_k$. We can apply this formula to $*\alpha$ to compute $(D(*\alpha), T(*\alpha))$. The formula we get is just the above formula with the range of the indices changed by their complementary; that is, one one hand we get terms of the form

$$\int_{\Gamma\backslash G}\langle Y_j\alpha_{i_1,\ldots,i_r},\rho(Y_j)\alpha_{i_1,\ldots,i_r}\rangle_V d\mu_G, \quad j\in\{i_1,\ldots,i_r\},$$

and on the other hand terms of the form

$$(-1)^{j+k}\int_{\Gamma\backslash G}\langle Y_{i_k}\alpha_{i_1,\ldots,\widehat{i_j},\ldots,i_k,\ldots,i_r},\rho(Y_{i_j})\alpha_{i_1,\ldots,i_j,\ldots,\widehat{i_k},\ldots,i_r}\rangle d\mu_G,$$

for $i_j \neq i_k$. By Lemma 1.9, this last term is the opposite of 7. Hence, it suffices to prove that for every $Y\in\mathfrak{m}$, and $f\in\mathcal{C}(\Gamma\backslash G;V)$, we have

$$\int_{\Gamma\backslash G}\langle Yf,\rho(Y)f\rangle_V d\mu_G = 0.$$

But it is also an immediate consequence of Lemma 1.9 and the symmetry of $\rho(Y)$. The lemma now follows from the fact that $(D^*\alpha, T^*\alpha) = (D(*\alpha), T(*\alpha))$.

□

Corollary 1.10 (Matsushima-Murakami formula). *Assume the inner product on V is symmetric respect to the action of $\mathfrak{m}$. Then*

$$\Delta_\rho = \Delta + H_\rho,$$

where $\Delta = DD^* + D^*D$, and $H_\rho = TT^* + T^*T$.

Proof. We have $\Delta_\rho = D_\rho D_\rho^* + D_\rho^* D_\rho = \Delta + H_\rho + S$, and Lemma 1.8. □

Let's denote by $\mathbf{T}$, $\mathbf{T}^*$, $\mathbf{H}_\rho$ the restriction to $V\otimes\bigwedge^p\mathfrak{m}^*$ of T, T^* and H_ρ respectively. Since T is an operator of degree zero, essentially all information of T, T^* and H_ρ is contained in $\mathbf{T}$, $\mathbf{T}^*$, $\mathbf{H}_\rho$. In particular, H_ρ is positive definite if and only $\mathbf{H}_\rho$ is so.

Proposition 1.11. *Let $\alpha\in V\otimes\bigwedge^p\mathfrak{m}^*$. Then we have,*

$$(\mathbf{H}_p\alpha)_{i_1,\ldots,i_r} = \sum_{j=1}^m \rho(Y_j)^2\alpha_{i_1,\ldots,i_r} + \sum_{k=1}^r\sum_{j=1}^m(-1)^{k+1}\rho([Y_{i_k},Y_j])\alpha_{j,i_1,\ldots,\widehat{i_k},\ldots,i_r}$$

Proof. Put $\beta_{i_1,\dots,i_{r+1}} = (T\alpha)_{i_1,\dots,i_{r+1}}$ and $\gamma_{i_1,\dots,i_{r-1}} = (T^*\alpha)_{i_1,\dots,i_{r-1}}$. Then, on one hand we have

$$\begin{aligned}(TT^*\alpha)_{i_1,\dots,i_r} &= \sum_{k=1}^{r}(-1)^{k+1}\rho(Y_{i_k})\gamma_{i_1,\dots,\widehat{i_k},\dots,i_r}\\ &= \sum_{k=1}^{r}(-1)^{k+1}\rho(Y_{i_k})\sum_{j=1}^{m}\rho(Y_j)\alpha_{j,i_1,\dots,\widehat{i_k},\dots,i_r}.\end{aligned}$$

and on the other hand,

$$\begin{aligned}(T^*T\alpha)_{i_1,\dots,i_r} &= \sum_{j=1}^{m}\rho(Y_j)\beta_{j,i_1,\dots,i_r}\\ &= \sum_{j=1}^{m}\rho(Y_j)\big(\rho(Y_j)\alpha_{i_1,\dots,i_r} + \sum_{k=1}^{r}(-1)^k\rho(Y_{i_k})\alpha_{j,i_1,\dots,\widehat{i_k},\dots,i_r}\big).\end{aligned}$$

And the proposition follows. □

1.2 Proof of Theorem 0.1

We want to apply the criterion of Andreotti-Vesentini of Theorem 1.1. For this purpose, we will use Matsushima-Murakami's formula (Corollary 1.10) for the representation of $\mathrm{SL}_2(\mathbf{C})$. Since for every compactly supported 1-form α

$$(\Delta(\alpha),\alpha) = (D(\alpha),D(\alpha)) + (D^*(\alpha),D^*(\alpha)) \geq 0,$$

using Corollary 1.10, the criterion of Theorem 1.1 reduces to show that $(H_\rho(\alpha),\alpha) \geq c(\alpha,\alpha)$ for some uniform $c > 0$ and every compactly supported 1-form α.

Notice that since the linear operator H_ρ on 1-forms is induced from a linear operator $\mathbf{H}_\rho$ on $V \otimes \mathfrak{m}^*$, if $\mathbf{H}_\rho$ is positive definite, then there is a positive constant c so that $(H_\rho(\alpha),\alpha) \geq c(\alpha,\alpha)$ holds for every compactly supported one form α. The proof will follow from Lemma 1.12.

In order to apply Matsushima-Murakami's formula to the representations of $\mathrm{SL}_2(\mathbf{C})$, first we need to choose an orthonormal basis for $\mathfrak{su}(2)$ respect to the Killing form (in fact, respect to a constant multiple of it). Let's define

$$X_1 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

Then (X_1,X_2,X_3) is an orthonormal basis for $\mathfrak{su}(2)$. The orthogonal complement to $\mathfrak{su}(2)$ with respect to the Killing form is given by $Y_k = \mathbf{i}X_k$, for $k = 1,2,3$. On the other hand, we have $[X_i,X_{i+1}] = 2X_{i+2}$, for $i = 1,2,3$, where the indices are taken modulo 3.

Lemma 1.12. *For* $\mathfrak{sl}(2,\mathbf{C})$ *and a* complex *representation* $\rho\colon \mathfrak{sl}(2,\mathbf{C}) \to$ *End*(V) *without trivial factors, the operator* $\mathbf{H}_\rho$ *is positively defined on degree* 1.

Proof. Since $\mathbf{H}_\rho = \mathbf{T}_\rho\mathbf{T}_\rho^* + \mathbf{T}_\rho^*\mathbf{T}_\rho$, to show that $\mathbf{H}_\rho$ is positive definite is equivalent to show that its kernel is trivial. Let $\alpha \in V \otimes \mathfrak{m}^*$. We have $\alpha = \sum_{i=1}^3 \alpha_i \otimes Y^i$, with $\alpha_i \in V$. Assume $\mathbf{H}_\rho\alpha = 0$. Then $\mathbf{T}_\rho\alpha = 0$ must vanish too, and from Proposition 1.4 (2) we obtain

$$0 = (\mathbf{T}_\rho\alpha)(Y_i, Y_j) = \rho(Y_i)\alpha_j - \rho(Y_j)\alpha_i, \quad i, j = 1, 2, 3. \tag{8}$$

Proposition 1.11 yields

$$(\mathbf{H}_\rho\alpha)(Y_j) = \sum_{k=1}^3 \big(\rho(Y_k)^2\alpha_j + \rho([Y_j, Y_k])\alpha_k\big).$$

Taking the indices modulo 3, and using the Lie algebra relations, we get

$$\begin{aligned}
\sum_{k=1}^3 \rho([Y_j, Y_k])\alpha_k &= \rho([Y_j, Y_{j+1}])\alpha_{j+1} + \rho([Y_j, Y_{j+2}])\alpha_{j+2} \\
&= 2\big(\rho(-X_{j+2})\alpha_{j+1} + \rho(X_{j+1})\alpha_{j+2}\big) \\
&= 2\mathbf{i}\big(\rho(Y_{j+2})\alpha_{j+1} - \rho(Y_{j+1})\alpha_{j+2}\big).
\end{aligned}$$

Notice that in the last equality we have used the complex structure. Hence, using 8, we get $(\mathbf{H}_\rho\alpha)(Y_j) = \sum_{k=1}^3 \rho(Y_k)^2\alpha_j$, and then

$$\begin{aligned}
0 = \langle \mathbf{H}_\rho\alpha, \alpha\rangle &= \sum_{j=1}^3 \Big\langle \sum_{k=1}^3 \rho(Y_k)^2\alpha_j, \alpha_j \Big\rangle \\
&= \sum_{j,k=1}^3 \big\langle \rho(Y_k)\alpha_j, \rho(Y_k)\alpha_j \big\rangle,
\end{aligned}$$

that implies $\rho(Y_j)\alpha_k = 0$ for $j, k = 1, 2, 3$. Hence, for a fixed k, we have $\rho(V)\alpha_k = 0$ for every $V \in \mathfrak{sl}(2,\mathbf{C})$. Since we are assuming that ρ has no trivial factors, we get $\alpha_k = 0$ for all k, as we wanted to prove. ☐

Remark. Since $\mathfrak{m}^* \cong \bigwedge^2 \mathfrak{m}^*$, the same proof holds true in degree 2.

2 Cohomology of the ends

Assume that M is a noncompact orientable hyperbolic manifold with finite topology, in particular it is the interior of a compact manifold with boundary $\partial\overline{M}$.

To go on, we need Poincaré duality with coefficients. This requires a nondegenerate invariant $\mathbf{C}$-bilinear pairing on any $\mathrm{SL}(2,\mathbf{C})$-module V.

Lemma 2.1. *For any* $\mathrm{SL}(2, \mathbf{C})$*-module* V *there exists a nondegenerate* $\mathbf{C}$*-bilinear invariant pairing*

$$\phi : V \times V \to \mathbf{C},$$

Proof. The $\mathrm{SL}(2, \mathbf{C})$-module V decomposes as direct sum of irreducible modules. In addition, irreducible modules are classified by its dimension over $\mathbf{C}$. Let V_n denote the n-dimensional irreducible $\mathrm{SL}(2, \mathbf{C})$, so that $V_1 \cong \mathbf{C}$ is the trivial representation and $V_2 \cong \mathbf{C}^2$ the standard one. The module V_n is the $(n-1)$-symmetric power of the standard one $V_2 \cong \mathbf{C}^2$. It follows from this classification that $V_n^* \cong V_n$, hence $V^* \cong V$ for every $\mathrm{SL}(2, \mathbf{C})$-module V. Thus there is an invariant bilinear pairing, by composing the isomorphism $V \times V \cong V^* \times V$ with the natural coupling between V^* and V. □

From this lemma we get (cf. [12, Sec. 2.2]):

Corollary 2.2. *Poincaré duality with coefficients in* E_ρ *holds true.*

Lemma 2.3. *Let* ρ_n *denote the irreducible* n*-dimensional representation. If* $n \geq 2$*, then*

$$\dim H^1(M; E_{\rho_n}) = \frac{1}{2} \dim H^1(\partial \overline{M}; E_{\rho_n}).$$

Proof. The map $H^1(M, \partial \overline{M}; E_{\rho_n}) \to H^1(M; E_{\rho_n})$ is zero by Theorem 0.1. So is $H^2(M, \partial \overline{M}; E_{\rho_n}) \to H^2(M; E_{\rho_n})$, because the proof of Lemma 1.12 in dimension 1 also works in dimension 2, as $\bigwedge^2 \mathfrak{m}^* = \mathfrak{m}^*$. Hence in the long exact sequence of the pair $(M, \partial \overline{M})$ we get a short exact sequence

$$0 \to H^1(M; E_{\rho_n}) \to H^1(\partial \overline{M}; E_{\rho_n}) \to H^2(\overline{M}, \partial M; E_{\rho_n}) \to 0.$$

Since $\dim H^1(M; E_{\rho_n}) = \dim H^2(\overline{M}, \partial M; E_{\rho_n})$ by Poincaré duality, this proves the lemma. □

To analyse the cohomology of $\partial \overline{M}$ we proceed separately on each connected component of $\partial \overline{M}$, i.e. on each end of M. We distinguish two cases, according to whether the end is a cusp (the end has finite volume and the component of $\partial \overline{M}$ is a torus) or not (the end has infinite volume and the component of $\partial \overline{M}$ has genus ≥ 2).

Assume first that that the end is a cusp $U \cong T^2 \times [0, +\infty)$, and for homotopy purposes, it may be easier to consider the compact 2-torus $T^2 \simeq U$. The fundamental group $\pi_1(U) = \pi_1(T^2) \cong \mathbf{Z} \oplus \mathbf{Z}$ is a peripheral subgroup of $\pi_1(M)$. The holonomy representation maps it to a parabolic subgroup, hence we may assume that every element in $\pi_1(T^2)$ is mapped to

$$\pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Definition. A lift to $\mathrm{SL}(2,\mathbf{C})$ of the holonomy restricted to $\pi_1(T^2)$ is called *positive* if the sign is $+$ for every element of $\pi_1(T^2)$.

When n is even, the positivity or not of the lift will determine the invariant subspaces of V_n, the n-dimensional irreducible representation of $\mathrm{SL}(2,\mathbf{C})$.

Definition. The *invariant subspace* V_n is the subspace consisting of elements of V_n that are fixed by $\pi_1(T^2)$:

$$V_n^{\pi_1(T^2)} = \{v \in V_n \mid \rho_n(\gamma)v = v, \text{ for all } \gamma \in \pi_1(T^2)\}.$$

We shall need to fix a basis for V_n. Let $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, so that $\{e_1, e_2\}$ is the standard basis for $V_2 = \mathbf{C}^2$. Thus

$$\{e_1^{n-1}, e_1^{n-2}e_2, \ldots, e_2^{n-1}\}$$

is a basis for $V_n = \mathrm{Sym}^{n-1}(V_2)$.

Lemma 2.4. *The invariant subspace is:*

$$V_n^{\pi_1(T^2)} = \begin{cases} 0 & \textit{for } n \textit{ even and a nonpositive lift;} \\ \mathbf{C} & \textit{for } n \textit{ even and a positive lift;} \\ \mathbf{C} & \textit{for } n \textit{ odd.} \end{cases}$$

Proof. After conjugation, elements of $\pi_1(T^2)$ have holonomy

$$\pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbf{C}).$$

Let $\{e_1, e_2\}$ denote the standard basis for $\mathbf{C}^2$. The previous matrix maps $e_1^{n-i-1}e_2^i$ to $(\pm 1)^{n-1}e_1^{n-i-1}(e_2 + \tau e_1)^i$, and it follows easily that there is no invariant subspace when n is even and the lift is nonpositive or it is generated by e_1^{n-1} otherwise. $\square$

Corollary 2.5. *For a cusp $U \cong T^2 \times [0, +\infty)$,*

$$H^1(U; E_n) = \begin{cases} 0 & \textit{for } n \textit{ even and a nonpositive lift;} \\ \mathbf{C}^2 & \textit{for } n \textit{ even and a positive lift;} \\ \mathbf{C}^2 & \textit{for } n \textit{ odd.} \end{cases}$$

Proof. Since $H^0(U; E_n) \cong H^0(T^2; E_n) \cong V_n^{\pi_1(T^2)}$, by Lemma 2.4 we get

$$H^0(T^2; E_n) = \begin{cases} 0 & \text{for } n \text{ even and a nonpositive lift;} \\ \mathbf{C} & \text{otherwise.} \end{cases}$$

Then by Poincaré duality and Euler characteristic arguments,

$$\dim H^2(T^2; E_n) = \dim H^0(T^2; E_n)$$

and

$$\dim H^1(T^2; E_n) = 2 \dim H^0(T^2; E_n).$$

$\square$

Now we discuss the case where the end U has infinite volume. Then $U \cong F_g \times [0, +\infty)$, where F_g is a surface of genus $g \geq 2$.

Lemma 2.6. *The invariant subspace of an infinite volume end is trivial:*

$$V_n^{\pi_1(F_g)} = 0$$

Proof. There exist at least one element $\gamma \in \pi_1(F_g)$ such that its holonomy is non parabolic. Up to conjugation, it is

$$\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

for some $\lambda \in \mathbf{C}$, with $|\lambda| > 1$. This means that the vectors e_1 and e_2 of the standard basis for $\mathbf{C}^2$ are eigenvectors. Since V_n is the $(n-1)$-symmetric power of $\mathbf{C}^2$, for n even V_n has no proper γ-invariant subspace, and for n odd, it is the line generated by $e_1^{\frac{n-1}{2}} e_2^{\frac{n-1}{2}}$. Any other matrix of $\mathrm{SL}_2(\mathbf{C})$ that fixes $e_1^{\frac{n-1}{2}} e_2^{\frac{n-1}{2}}$ is either diagonal or antidiagonal (zero entries in the diagonal). Antidiagonal matrices have trace zero, hence they have order four, so they cannot occur because the holonomy is faithful. Also, the holonomy representation of any element in $\pi_1(F)$ that does not commute with γ is never diagonal. $\square$

By an Euler characteristic argument and by Poincaré duality as in Corollary 2.5, we have:

Corollary 2.7. *For an infinite volume end $U \cong F_g \times [0, +\infty)$,*

$$\dim_{\mathbf{C}} H^1(U; E_n) = (2g-2)n.$$

Proposition 2.8. *If M has k cusps and l ends of infinite volume of genus $g_1, \ldots, g_l$, then for $n \geq 2$,*

$$\dim_{\mathbf{C}}(H^1(M; E_n)) = \sum_{i=1}^{l} n(g_i - 1) + \begin{cases} k_0 & \textit{for } n \textit{ even;} \\ k & \textit{for } n \textit{ odd.} \end{cases}$$

where $0 \leq k_0 \leq k$ is the number of cusps for which the lift of the holonomy is positive.

Proof. Apply Lemma 2.3 and Corollaries 2.5 and 2.7. $\square$

3 Infinitesimal Rigidity

Here we prove Theorem 0.5, that we restate.

Theorem 3.1. *Let M be a complete hyperbolic 3-manifold that is topologically finite. If $\partial\overline{M}$ is the union of k tori and l surfaces of genus $g_1, \ldots, g_l \geq 2$, and $n \geq 2$, then*

$$H^1(M; E_{\mathrm{Ad}\,\circ\rho_n}) \cong \mathbf{C}^{k(n-1)+\sum(g_i-1)(n^2-1)}.$$

In particular, if M is closed then $H^1(M; E_{\mathrm{Ad}\,\circ\rho_n}) = 0$. In addition, all nontrivial elements in $H^1(M; E_{\mathrm{Ad}\,\circ\rho_n})$ are nontrivial in $\partial\overline{M}$ and have no L^2 representative.

Proof. As $\mathrm{SL}(2,\mathbf{C})$-modules, we have

$$\mathfrak{sl}(n,\mathbf{C}) \oplus \mathbf{C} \cong \mathfrak{gl}(n,\mathbf{C}) \cong V_n \otimes V_n^* \cong \mathrm{Sym}^{n-1}V \otimes \mathrm{Sym}^{n-1}V.$$

On the other hand,

$$\mathrm{Sym}^{n-1}V \otimes \mathrm{Sym}^{n-1}V \cong \mathrm{Sym}^{2(n-1)}V \oplus \mathrm{Sym}^{2(n-2)}V \oplus \cdots \oplus \mathrm{Sym}^2 V \oplus \mathbf{C}$$

(cf. [10, Ex. 11.11]). Therefore, we get

$$\mathfrak{sl}(n,\mathbf{C}) \cong \mathrm{Sym}^{2(n-1)}V \oplus \mathrm{Sym}^{2(n-2)}V \oplus \cdots \oplus \mathrm{Sym}^2 V.$$

Hence,

$$H^1(M, E_{\mathrm{Ad}\,\circ\rho_n}) \cong H^1(M; E_{\rho_{2n-1}}) \oplus H^1(M; E_{\rho_{2n-3}}) \oplus \cdots \oplus H^1(M; E_{\rho_3}). \quad (9)$$

The proof of the theorem now follows from this isomorphism and Proposition 2.8. $\square$

Next we want to prove Theorem 0.6. See [19] for basic results about representation and character varieties. The variety of representations of $\pi_1(M)$ in $\mathrm{SL}(n,\mathbf{C})$ is

$$R(M, \mathrm{SL}(n,\mathbf{C})) = \hom(\pi_1(M), \mathrm{SL}(n,\mathbf{C})).$$

Since $\pi_1(M)$ is finitely generated, this is an algebraic affine set. The group $\mathrm{SL}(n,\mathbf{C})$ acts by conjugation on $R(M,\mathrm{SL}(n,\mathbf{C}))$ algebraically, and the quotient in the algebraic category is the variety of characters:

$$X(M, \mathrm{SL}(n,\mathbf{C})) = R(M, \mathrm{SL}(n,\mathbf{C}))/\!/\, \mathrm{SL}(n,\mathbf{C}).$$

For a representation $\rho \in R(M, \mathrm{SL}(n,\mathbf{C}))$ its character is the map

$$\begin{array}{rcl} \chi_\rho\colon \pi_1(M) & \to & \mathbf{C} \\ \gamma & \mapsto & \mathrm{trace}(\rho(\gamma)) \end{array}$$

The projection $R(M,\mathrm{SL}(n,\mathbf{C})) \to X(M,\mathrm{SL}(n,\mathbf{C}))$ maps each representation ρ to its character χ_ρ.

Weil's construction gives a natural isomorphism between the Zariski tangent space to a representation $T^{Zar}_{\rho} R(M, \mathrm{SL}(n, \mathbf{C}))$ and $Z^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho})$, the space of group cocycles valued in the lie algebra $\mathfrak{sl}(n, \mathbf{C})$, which as $\pi_1(M)$-module is also written as $V_{\mathrm{Ad} \circ \rho}$. Namely, $Z^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho})$ is the set of maps $d\colon \pi_1(M) \to V_{\mathrm{Ad} \circ \rho}$ that satisfy the cocycle relation

$$d(\gamma_1 \gamma_2) = d(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)} d(\gamma_2), \qquad \forall \gamma_1, \gamma_2 \in \pi_1(M).$$

Notice that $R(M, \mathrm{SL}(n, \mathbf{C}))$ may be a non reduced algebraic set, so the Zariski tangent space may be larger than the Zariski tangent space of the underlying algebraic variety.

The space of coboundaries $B^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho})$ is the set of cocycles that satisfy $d(\gamma) = \mathrm{Ad}_{\rho(\gamma)} m - m$ for all $\gamma \in \pi_1(M)$ and for some fixed $m \in V_{\mathrm{Ad} \circ \rho}$. The space of coboundaries is the tangent space to the orbit by conjugation, so under some reasonable hypothesis the cohomology may be identified with the tangent space of the variety of characters. The group cohomology of $\pi_1(M)$

$$H^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho}) = Z^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho}) / B^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho})$$

is naturally identified to $H^1(M; E_{\mathrm{Ad} \circ \rho})$.

Definition. A representation $\rho\colon \pi_1(M) \to \mathrm{SL}(n, \mathbf{C})$ is *semisimple* if every subspace of $\mathbf{C}^n$ invariant by $\rho(\pi_1(M))$ has an invariant complement.

Thus a semisimple representation decomposes as direct sum of simple representations, where simple means without proper invariant subspaces.

The following summarises the relation between tangent spaces and cohomology. See [19] for a proof.

Proposition 3.2. *Let $\rho \in R(M, \mathrm{SL}(n, \mathbf{C}))$.*

1. *There is a natural isomorphism*

$$Z^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho}) \cong T^{Zar}_{\rho} R(M, \mathrm{SL}(n, \mathbf{C})).$$

2. *If ρ is semisimple, then it induces an isomorphism*

$$H^1(\pi_1(M), V_{\mathrm{Ad} \circ \rho}) \cong T^{Zar}_{\rho} X(M, \mathrm{SL}(n, \mathbf{C})).$$

3. *If ρ is semisimple and a smooth point of $R(M, \mathrm{SL}(n, \mathbf{C}))$, then its character χ_ρ is a smooth point of $X(M, \mathrm{SL}(n, \mathbf{C})$.*

A point in an algebraic affine set is smooth iff it has the same dimension that its Zariski tangent space. So to prove smoothness we need to compute these dimensions.

Lemma 3.3. *Let ρ_n be as in Theorem 0.6, and T^2 a component of $\partial\overline{M}$ corresponding to a cusp. Then the restriction of ρ_n to $\pi_1(T^2)$ is a smooth point of $R(T^2, \mathrm{SL}(n, \mathbf{C}))$.*

Proof. Knowing that $\dim R(T^2, \mathrm{SL}(n, \mathbf{C})) \leq \dim Z^1(T^2, V_{\mathrm{Ad}\circ\rho})$, we want to show that equality of dimensions hold. Before the cocycle space, we first compute the dimension of the cohomology group. By Equation 9 in the proof of Theorem 3.1:

$$\dim H^1(T^2, E_{\mathrm{Ad}\circ\rho}) = \dim H^1(T^2, E_{\rho_{2n-1}}) + \dim H^1(T^2, E_{\rho_{2n-3}}) + \cdots + \dim H^1(T^2, E_{\rho_3}).$$

Hence, by Corollary 2.5,

$$\dim H^1(T^2, E_{\mathrm{Ad}\circ\rho}) = 2(n-1).$$

We apply the same decomposition for computing the dimension of the coboundary space. It is the sum of terms $\dim B^1(T^2, E_{\rho_k})$, for k odd from 3 to $2n-1$. Since we have an exact sequence

$$0 \to V_k^{\pi_1(T^2)} \to V_k \to B^1(T^2, E_{\rho_k}) \to 0,$$

$\dim B^1(T^2, E_{\rho_k}) = k - \dim V_k^{\pi_1(T^2)} = k - 1$, by Lemma 2.4. Thus

$$\dim B^1(T^2, E_{\mathrm{Ad}\circ\rho}) = (2n-2) + (2n-4) + \cdots + 2 = n^2 - n.$$

Hence as $H^1(T^2, E_{\mathrm{Ad}\circ\rho}) = Z^1(T^2, E_{\mathrm{Ad}\circ\rho})/B^1(T^2, E_{\mathrm{Ad}\circ\rho})$, we have:

$$\dim Z^1(T^2, E_{\mathrm{Ad}\circ\rho}) = \dim H^1(T^2, E_{\mathrm{Ad}\circ\rho}) + \dim B^1(T^2, E_{\mathrm{Ad}\circ\rho}) = n^2 + n - 2.$$

Now we look for a lower bound of $\dim R(T^2, \mathrm{SL}(n, \mathbf{C}))$. Fix $\{\gamma_1, \gamma_2\}$ a generating set of $\pi_1(T^2)$. The representation ρ_n restricted to $\pi_1(T^2)$ has eigenvalues equal to ± 1. By deforming the representation of $\pi_1(T^2)$ to $\mathrm{SL}(2, \mathbf{C})$, and by composing it with the representation of $\mathrm{SL}(2, \mathbf{C})$ to $\mathrm{SL}(n, \mathbf{C})$, there exists a representation $\rho' \in R(T^2, \mathrm{SL}(n, \mathbf{C}))$ arbitrarily close to ρ_n such that all eigenvalues of $\rho'(\gamma_1)$ are different, in particular $\rho'(\gamma_1)$ diagonalises. Now, to find deformations of ρ', notice that $\rho'(\gamma_1)$ can be deformed with $n^2 - 1 = \dim(\mathrm{SL}(n, \mathbf{C}))$ parameters, and having all eigenvalues different is an open condition. As $\rho'(\gamma_2)$ has to commute with $\rho'(\gamma_1)$, it has the same eigenspaces, but one can still chose $n-1$ eigenvalues for $\rho'(\gamma_2)$. This proves that the dimension of some irreducible component of $R(T^2, \mathrm{SL}(n, \mathbf{C}))$ that contains ρ_n is at least

$$n^2 - 1 + n - 1 = n^2 + n - 2.$$

As this is $\dim Z^1(T^2, E_{\mathrm{Ad}\circ\rho})$, it is a smooth point. □

Proof of Theorem 0.6. Using Proposition 3.2, we just prove that ρ_n is a smooth point of the variety of representations.

Given a Zariski tangent vector $v \in Z^1(M, V_{\mathrm{Ad}\circ\rho})$, we have to show that it is integrable, i.e. that here is a path in the variety of representations whose tangent vector is v. For this, we use the algebraic obstruction theory, see [12, 14]. There exist an infinite sequence of obstructions that are cohomology classes in $H^2(M, V_{\mathrm{Ad}\circ\rho})$, each obstruction being defined only if the previous one vanishes. These are related to the analytic expansion in power series of a deformation of a representation, and to Kodaira's theory of infinitesimal deformations. Our aim is to show that this infinite sequence vanishes. This gives a formal power series, that does not need to converge, but this is sufficient for v to be a tangent vector by a theorem of Artin [3] (see [14] for details).

We do not give the explicit construction of these obstructions, we just use that they are natural and that they live in the second cohomology group. We work with the long exact sequence of the pair $(M, \partial\overline{M})$:

$$\cdots \to H^1(M; E_{Ad\circ\rho}) \to H^1(\partial\overline{M}; E_{Ad\circ\rho}) \to H^2(M, \partial\overline{M}; E_{Ad\circ\rho}) \to \cdots$$

By Corollary 0.3, $H^1(M; E_{Ad\circ\rho}) \to H^1(\partial\overline{M}; E_{Ad\circ\rho})$ is injective, and by Poincaré duality and Lemma 2.3, $H^1(\partial\overline{M}; E_{Ad\circ\rho}) \to H^2(M, \partial\overline{M}; E_{Ad\circ\rho})$ is surjective. Also $H^0(M; E_{Ad\circ\rho}) = 0$, and by Poincaré duality again, $H^3(M, \partial\overline{M}; E_{Ad\circ\rho}) = 0$. Thus we get an isomorphism:

$$H^2(M; E_{Ad\circ\rho}) \cong H^2(\partial\overline{M}; E_{Ad\circ\rho}) \tag{10}$$

Now, $H^2(\partial\overline{M}; E_{Ad\circ\rho})$ decomposes as the sum of the connected components of $\partial\overline{M}$. If F_g has genus $g \geq 2$ then $H^2(F_g; E_{Ad\circ\rho}) = 0$. Thus, only the components of $\partial\overline{M}$ that are tori appear in $H^2(\partial\overline{M}; E_{Ad\circ\rho})$. By Lemma 3.3 and naturality, the obstructions vanish when restricted to $H^2(T^2; E_{Ad\circ\rho})$, hence they vanish in $H^2(M; E_{Ad\circ\rho})$ by the isomorphism (10). □

4 Lifting the holonomy representation

Proposition 4.1 ([9])**.** *The holonomy representation of a hyperbolic 3-manifold M lifts to $\mathrm{SL}_2(\mathbf{C})$. In addition, there is a natural bijection between the set of lifts and the set of spin structures.*

This is proved in Section 2 of [9]. Essentially the idea is that a spin structure on M has a section, because M is parallelizable, and this section lifts to a equivariant section of the spin bundle on the universal covering of M. Identifying the universal covering of M with $\mathbf{H}^3$, the spin bundle corresponds to $\mathrm{SL}_2(\mathbf{C})$, and equivariance of the section gives the lifted representation of $\pi_1(M)$ in $\mathrm{SL}_2(\mathbf{C})$. Notice that on both sets, the set of spin structures and the set of lifts, there is a simply transitive action of $H^1(M; \mathbf{Z}/2\mathbf{Z})$. We view

elements in $H^1(M;\mathbf{Z}/2\mathbf{Z})$ as homomorphisms $\pi_1(M) \to \mathbf{Z}/2\mathbf{Z}$ that describe the difference between signs of two different lifts.

Assume that M has k cusps, and choose $\gamma_1, \ldots, \gamma_k \in \pi_1(M)$ k elements so that each γ_i is represented by a simple closed curve in one of the torus of the cusp, and different curves go to different cusps.

Lemma 4.2. *For any choice of curves as above, there exists a lift $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbf{C})$ of the holonomy representation such that*

$$\mathrm{trace}(\rho(\gamma_i)) = -2.$$

for $i = 1, \ldots, k$.

Proof. We denote the peripheral torus by $T_1^2, \ldots, T_k^2$. Let $\mu_i \in \pi_1(T_i^2)$ be represented by a simple closed curve intersecting γ_i in one point, so that γ_i and μ_i generate $\pi_1(T_i^2)$. We can replace γ_i by $\gamma_i \mu_i^{2n_i}$, for some n_i sufficiently large, as multiplying by an even power of μ_i does not change the sign of the trace.

We chose the n_i sufficiently large so that Thurston's hyperbolic Dehn filling applies to these slopes. More precisely, we require that there is a continuous path of cone manifold structures with cone angle $\alpha \in [0, 2\pi]$, so that $\alpha = 0$ is the complete structure on M and $\alpha = 2\pi$ is the filled manifold (cf. [24, 16]). Now we chose the lift of the hyperbolic structure on the filled manifold, using Culler's theorem [9], and consider the induced lifts corresponding to changing continuously the cone angle. The map $X(M, SL(2,\mathbf{C})) \to X(M, PSL(2,\mathbf{C}))$ is a local homeomorphism except at characters of reducible representations or representations that preserve a (unoriented) geodesic of $\mathbf{H}^3$ [13]. Thus we get a continuous path of representations in $X(M, SL(2,\mathbf{C}))$ parametrized by the cone angle $\alpha \in [0, 2\pi]$, cf. [9, Thm. 4.1].

The holonomy of γ_i is conjugate to

$$\pm \begin{pmatrix} \exp(i\,\alpha/2) & 0 \\ 0 & \exp(-i\,\alpha/2) \end{pmatrix}$$

and its trace is $\pm 2\cos(\alpha/2)$. The sign $\pm$ must be constant by continuity. This is clear when $\alpha \neq \pi$ because then the trace is nonzero. When $\alpha = \pi$, we use the local rigidity theorem of [15, 26], that says that this path is locally parametrized by α, and since the derivative of $\pm 2\cos(\alpha/2)$ at $\alpha = \pi$ is $\pm \sin(\pi/2) = \pm 1$, the trace is monotonic on α when $\alpha = \pi$.

Finally, since we have chosen a lift that is trivial on γ when $\alpha = 2\pi$, the choice of sign is

$$-2\cos(\alpha/2),$$

and when $\alpha = 0$ we get the result. □

We obtain the following well known result, proved by Calegari in [7], that applies for instance to the longitude of a knot.

Corollary 4.3. *Let γ be a simple closed curve in a torus of $\partial\overline{M}$ homotopically nontrivial. If γ is homologous to zero in $H_1(M;\mathbf{Z}/2\mathbf{Z})$, then, for every lift $\varphi\colon \pi_1(M)\to \mathrm{SL}_2(\mathbf{C})$ of the holonomy representation,*

$$\mathrm{trace}(\varphi(\gamma)) = -2.$$

Proof. The proof follows from the fact that the sign of $\varphi(\gamma)$ cannot be changed by taking different lifts, and by applying Lemma 4.2. $\square$

Proof of Theorem 0.4. Since the inclusion in homology

$$H_1(U;\mathbf{Z}/2\mathbf{Z}) \to H_1(M;\mathbf{Z}/2\mathbf{Z})$$

has rank one, there exists a simple closed curve representing a nontrivial element in $H_1(T^2;\mathbf{Z}/2\mathbf{Z}) \cong H_1(U;\mathbf{Z}/2\mathbf{Z})$ which is $\mathbf{Z}/2\mathbf{Z}$-homologous to zero in M. Thus Corollary 4.3 applies here and every lift of the holonomy restricted to the peripheral group is nonpositive. Hence Theorem 0.4 follows from Proposition 2.8. $\square$

A Some results on principal bundles

Throughout this section P will denote a G-principal bundle over a manifold M.

Remark. We will follow the convention that the action of G is on the right.

Assume we have a connection on P with connection form $\omega \in \Omega^1(P;\mathfrak{g})$. This connection defines a horizontal vector bundle H on P. The differential of the bundle projection $\pi_P\colon P\to M$ is an isomorphism when restricted to H. Hence, given $X_p \in TM$ and $u\in\pi_P^{-1}(p)$, there exists a unique $\tilde{X}_u \in H_u$ that is projected to X_p. The vector $\tilde{X}_u$ is called the horizontal lift of X_p at u. A vector field on P is called horizontal if it is tangent to H.

All these definitions can be extended in a natural way to the cotangent bundle, exterior powers, tensor powers, etc. Therefore, it makes sense to talk about horizontal forms, horizontal tensors, etc.

Let's recall a common construction. Let F be a differentiable manifold on which G acts on the left. The associated bundle, denoted by $P\times_G F$, is the quotient of $P\times V$ by the diagonal right action of G (i. e. if $(u,x)\in P\times F$, then $(u,x)\cdot g = (ug, g^{-1}x)$). The space $P\times_G F$ has in a natural way a structure of fiber bundle over M with typical fiber F.

Observation. The definition of $P \times_G F$ allows us to interpret a point u in P as an isomorphism between F and the fiber of $P \times_G F$ at $\pi_P(u)$. Let's say, if π denotes the quotient map $P \times V \to P \times_G F$, then $\pi(u, \cdot)$ is an isomorphism. Note that $\pi(u \cdot g, x) = \pi(u, gx)$.

We can generalize the notion of associated bundle just "twisting F"; that is, we can take as a starting point an arbitrary bundle over P with typical fiber F, instead of just the product bundle $P \times F$. Let $\pi_Q \colon Q \to P$ be a bundle over P with typical fiber F. Assume that we have a fiber-preserving action (on the right) of G on Q that is compatible with the action on P (i. e. $\pi_Q(q \cdot g) = \pi_Q(q) \cdot g$). The quotient Q/G is in a natural way a fiber bundle over M with typical fiber F. In this case, a point $u \in P$ can be interpreted as an isomorphism between the fiber of Q at u, and the fiber of Q/G at $\pi(u)$.

Proposition A.1. *There is a canonical isomorphism between the space of G-equivariant sections of Q, and the space of sections of the associated bundle Q/G.*

Now we want to specialize all these things to the case $Q = \bigwedge^r H^* \otimes V$, where V is a fixed vector space. Let's fix a linear representation $\rho \colon G \to \mathrm{Aut}(V)$, in such a way that V becomes a left G-module. We then let act G on Q on the right as follows: if $\alpha_p \otimes w_p$ belongs to Q_p, then $(\alpha_p \otimes w_p) \cdot g = R^*_{g^{-1}} \alpha_p \otimes \rho(g)^{-1} w_p \in Q_{pg}$. Using horizontal lifts we can identify Q/G with $\bigwedge^r T^*M \otimes E$. More precisely, let $p \in M$, $u \in \pi^{-1}(p)$, and $\mathrm{H}_u \colon T_pM \to H_u$ the horizontal lift map. Then, if we interpret u an isomorphism between V and E_p, we obtain the isomorphism $\varphi_u \colon \mathrm{H}^*_u \otimes u \colon Q_u \to \bigwedge^r T_p{*}M \otimes E_p$. Since horizontal lift and u commute with the action of G, we have $\varphi_u(v) = \varphi_{ug}(vg)$, for all $v \in Q$. Therefore, we get an isomorphism φ between Q/G and $\bigwedge^r T^*M \otimes E$.

We will denote by $\Omega^*_{\mathrm{Hor}}(P; V)^G$ the space of horizontal V-valued differential forms over P that are G-equivariant, or, equivalently, the space of G-equivariant sections of the bundle $\bigwedge^r H^* \otimes V$.

Observation. A form α is horizontal if, and only if, it vanishes on vertical directions, that is, $i_X \alpha = 0$ for any vertical vector field X. Also, α is G-equivariant if, and only if, $R^*_g \alpha = \rho(g^{-1})\alpha$ for all $g \in G$. Therefore, $\alpha \in \Omega^r(P; V)$ belongs to $\Omega^r_{\mathrm{Hor}}(P; V)^G$ if, and only if,

$$R^*_g \alpha = \rho(g)^{-1}\alpha, \quad \text{per a tot } g \in G, \tag{11}$$

$$i_Y \alpha = 0, \quad \text{per a tot } Y \in \mathfrak{g}. \tag{12}$$

Note that we are identifying $\mathfrak{g}$ with the space of G-invariant vertical vectors over a fixed fiber of P.

The connection on P defines an exterior covariant differential on G-equivariant horizontal forms. Namely,

$$D\alpha = (d\alpha) \circ \pi_h, \quad \text{for} \quad \alpha \in \Omega^r_{\mathrm{Hor}}(P;V)^G$$

where π_h is projection on the horizontal distribution defined by the connection. On the other hand, a connection on P induces a connection on the vector bundle $P \times_\rho V$, and hence an exterior covariant differential d_ρ on $\Omega^r(M;E)$. It is easily verified that the canonical isomorphism between the spaces $\Omega^*_{\mathrm{Hor}}(P;V)^G$ and $\Omega^*(M;E)$, "commute" with exterior covariant differentiation (see [18, p. 76]).

Proposition A.2. *Let $\omega \in \Omega^1(P;\mathfrak{g})$ be the connection form of the connection defined on P. Then the following formula holds*

$$D\alpha = d\alpha + \rho(\omega) \wedge \alpha.$$

Remark. If $V_1, \dots, V_{p+1}$ are vector fields on P, by definition,

$$(\rho \wedge \alpha)(V_1, \dots, V_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \rho(Y_i)\big(\alpha(V_1, \dots, \widehat{V_i}, \dots, V_{p+1})\big).$$

Taking a base of V, $\rho(w)$ is just a matrix of 1-forms, α a column vector of p-forms, and the product $\rho(\omega) \wedge \alpha$ is just the product of a matrix by a vector.

Proof. We must prove the form $d\alpha + \rho(\omega) \wedge \alpha$ is horizontal, and that on horizontals vectors coincides with $D\alpha$. The second fact is obvious from the the definition of D and the fact that ω vanishes on horizontal vectors. Hence we only need to prove that $d\alpha + \rho(\omega) \wedge \alpha$ vanishes on vertical vectors. Let be X^* the fundamental vector field associated to $X \in \mathfrak{g}$, using Cartan's identity $(L^*_X = di^*_X + i^*_X d))$ we get $i^*_X(d\alpha + \rho(\omega) \wedge \alpha) = L^*_X \alpha - d(i^*_X \alpha) + \rho(X)\alpha)$. The infinitesimal version of the G-equivariance of α states that $L^*_X \alpha = -\rho(X)\alpha$. Then we conclude that $d\alpha + \rho(\omega) \wedge \alpha$ is vertical. $\square$

Now assume that M is a Riemannian manifold, and that we have a metric on the vector bundle $E = P \times_G V$. These metrics induce an inner product on the space of E-valued forms over M.

$$(\alpha, \beta) = \int_M \langle \alpha(x), \beta(x) \rangle_x \omega_M.$$

On the other hand, the Riemannian metric on M defines a metric tensor on the horizontal bundle H, in such a way that horizontal lifts are isometries. Also, the metric defined on E defines a metric on the trivial vector bundle $P \times V$. A right invariant volume form ω_G on G defines a right invariant

volume form along the fibers of P. Therefore, we can define an inner product on $\Omega^r_{\mathrm{Hor}}(P;V)^G$ by

$$(\tilde{\alpha},\tilde{\beta}) = \int_P \langle \tilde{\alpha}(u), \tilde{\beta}(u)\rangle_u \pi_P^*(\omega_M) \wedge \omega_G^*.$$

We want to study how the metrics defined on $\Omega^r(M;E)$ and $\Omega^r_{\mathrm{Hor}}(P;V)^G$ are related by the canonical isomorphism. However, this comparison doesn't make sense if G is not assumed to be compact (if $\alpha \in \Omega^r(M;E)$ has compact support, then the corresponding form $\tilde{\alpha}$ in $\Omega^r_{\mathrm{Hor}}(P;V)^G$ has compact support if, and only if, G is compact). From now on we will assume that G is compact. In order to avoid confusions we will denote G by K in this case. In this case we can simplify things a little bit. First, take a K-invariant metric on V, and use it to define a "constant" metric on $P \times V$. Since this metric is K-invariant, we get a metric on the vector bundle E. Under these hypothesis, we get a nice relation between these two metrics. In order to get this relation, we need the following Lemma.

Proposition A.3. *Let ω_M be a volume form on M, and ω_K a right invariant volume form on K. Denote by ω_K^* the right invariant volume form on the fibers of P defined by ω_K. If f is a function defined on P, then the function $\bar{f}(u) = \int_K f(ug)\omega_K$ is invariant along the fibers, and hence can be seen as a function on M. With these hypothesis, we have*

$$\int_P f(u)\pi_P^*(\omega_M) \wedge \omega_K^* = \int_M \bar{f}(x)\omega_M$$

Proof. Take an open set $U \subset M$ that trivializes P, and a trivializing map $\psi\colon U \times K \to \pi_P^{-1}(U)$. Let's denote by π_U and π_K the projection of $U \times K$ on the first and on the second factor respectively. We have $\omega_K^* = (\psi^*)^{-1}(\pi_K^*(\omega_K))$. The change of variable formula gives

$$\int_{\pi_P^{-1}(U)} f(u)\pi_P^*(\omega_M) \wedge \omega_K^* = \int_{U\times K} f(\psi(x,g))\pi_U^*(\omega_M) \wedge \pi_K^*(\omega_K).$$

By Fubini's Theorem, the last integral is,

$$\int_U (\int_K f(\psi(x,g))\omega_K)\omega_M = \int_U \bar{f}(x)\omega_M.$$

The result follows by taking a partition of unity subordinated to a trivializing open cover. □

The function $\langle \tilde{\alpha}(u), \tilde{\beta}(u)\rangle_V$ is constant along the fibers, and equals to $\langle \alpha(x), \beta(x)\rangle_x$, where $x = \pi_P(u)$. The above Lemma then implies the following Proposition.

Proposition A.4. *With the above notation,*

$$(\tilde{\alpha}, \tilde{\beta}) = \mu(K)(\alpha, \beta),$$

where μ denotes the measure defined by the volume form ω_K.

Consider the pairing

$$\begin{array}{ccc} \Omega^r_{\mathrm{Hor}}(P;V)^K \times \Omega^{m-r}_{\mathrm{Hor}}(P;V)^K & \longrightarrow & \mathbf{R} \\ (\alpha, \beta) & \longmapsto & \int_P (\alpha \wedge \beta) \wedge \omega_K, \end{array}$$

where the wedge product of a V-valued is defined using the usual wedge product on scalar-valued forms, and the inner product on V. On the other hand, the metric on the horizontal bundle, and the orientation we have on it, allow us to define a Hodge star operator on the space of horizontal forms,

$$*\colon \Omega^r_{\mathrm{Hor}}(P;V)^K \longrightarrow \Omega^{m-r}_{\mathrm{Hor}}(P;V)^K.$$

Note that we have $(\alpha, \beta) = \phi(\alpha, *\beta)$

Proposition A.5. *Let $T\colon \Omega^r_{\mathrm{Hor}}(P;V)^K \to \Omega^{r+k}_{\mathrm{Hor}}(P;V)^K$ be a linear operator that decreases supports. Assume we have a linear operator*

$$S\colon \Omega^{m-(r+k)}_{\mathrm{Hor}}(P;V)^K \to \Omega^{m-r}_{\mathrm{Hor}}(P;V)^K$$

such that $\phi(T\alpha, \beta) = \phi(\alpha, S\beta)$. Then, the formal adjoint of T is

$$T^* = (-1)^{r(m-r)} * S*\colon \Omega^{r+k}_{\mathrm{Hor}}(P;V)^K \to \Omega^r_{\mathrm{Hor}}(P;V)^K.$$

Proof. Let's denote $\Omega^r_{\mathrm{Hor}}(P;V)^K$ by M_r. We have the following commutative diagram,

$$\begin{tikzcd} M^*_{r+k} \arrow[r, "T^t"] & M^*_r \\ M_{r+k} \arrow[u] \arrow[r, "T^*"] & M_r \arrow[u] \end{tikzcd}$$

where the vertical arrows are the isomorphisms given by the metrics, T^t is the dual map of T, and T^* its adjoint. We can factor the metric isomorphism as $\phi(*,)$. We have the following commutative diagram

$$\begin{tikzcd} M^*_{r+k} \arrow[r, "T^t"] & M^*_r \\ M_{m-(r+k)} \arrow[u, "\phi(\cdot,)"] \arrow[r, "S"] & M_{m-r} \arrow[u, "\phi(\cdot,)"'] \\ M_{r+k} \arrow[u, "*"] \arrow[r, "T^*"] & M_r \arrow[u, "*"'] \end{tikzcd} .$$

The Proposition now follows from the fact that on degree r we have $*^{-1} = (-1)^{r(m-r)}*$.

□

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